Indian Statistical Institute
Bangalore Centre
B.Math Third Year 2014-2015
Second Semester

Semestral Examination Date :05.05.15

Statistics IV

Answer as much as you can. The maximum you can score is 120.
The notation used have their usual meaning unless stated otherwise.
State clearly the results you use.
Time :- 3 hours

1. Consider a study with three categorical variables X, Y and Z in which X and Y takes two values while Z takes K values.

   (a) Define conditional odds ratio θ_k^{XY}, $1 \leq k \leq K$.

   (b) Show that each of the following is a sufficient condition for θ_k^{XY} being independent of k.

   (i) X is conditionally independent of Z, given Y.

   (ii) Y is conditionally independent of Z, given X. [2 + 4 x 2 =13]

2. Suppose $X = (X_1, \cdots X_k)'$ follows multinomial distribution with parameters $(n, \pi_1, \cdots \pi_k)$.

   (a) Suppose each π is a function of $\theta_1, \cdots \theta_q$, $q < k$. Let $M = ((M_{ij}))$, $m_{ij} = (\sqrt{\pi_i})^{-1}\frac{\partial \pi_i}{\partial \theta_j}$.

   Show that the information matrix $= nM'M$.

   [Recall that the (s,t)th element of the information matrix is $E[\frac{\partial log(f(x,\theta))}{\partial \theta_s}\frac{\partial log(f(x,\theta))}{\partial \theta_t}]$, where $f(x,\theta)$ is the p.d.f of X].

   (b) Explain how you can test the hypothesis that each π_i is a function of $\theta_1, \cdots \theta_q, q < k$. State clearly the result you use without proof.

   (c) The distribution of four blood groups of individuals in a random sample from a population of two communities is given in the following table. Analyse the data in the following table to see if the proportions of the blood groups are equal for the different communities.

	Blood		group	
Community	O	A	B	AB
1	121	120	79	33
2	118	95	121	30

[8 + 3 + 5 =16]

3. Suppose $X_1, \cdots X_n$ is a random samples from a continuous distribution with median θ. Let R_i^a be the rank of $|X_i|$ and $T^+ = \sum_{X_i > 0} R_i^a$. Let $W_{ij} = (1/2)(X_i + X_j)$.

   (a) Show that under $H_0 : \theta = 0$, the distribution of T^+ is symmetric about $n(n+1)/4$.

   (b) Suppose $\theta = 0$. Let W^+ denote the number of positive W_{ij}'s. If (a) $X_i \neq 0$ and (b) $|X_i| \neq |X_j|$, then show that $W^+ = T^+$.

   (c) Show how you can find a confidence interval for θ using W_{ij}'s. [4 + 5 + 10 = 19]

4. Suppose $X_1, \cdots X_m$ and $Y_1, \cdots Y_n$ are independent random samples from continuous distributions with c.d.f $F(x)$ and $G(y) = F(y - \triangle)$ respectively. Describe the distribution of the following statistics under $H_0 : \triangle = 0$. Assume that m is odd and n is even.

   (a) $V_1 =$ the number of Y's greater than the largest X.

   (b) $V_2 =$ the number of Y's greater than the median of the combined sample of the $m + n$ observations. [4 + 6 =10]

5. (a) Consider a decision problem. Define (i) a minmax decision rule and (ii) a Bayes' decision rule w.r.t. a prior distribution π.

   When is a decision rule said to be admissible ?

   (b) Consider a decision problem with $\Theta = \{\theta_1, \theta_2\}$, set of actions $\mathcal{A} = \{a_1, \cdots a_4\}$ and the loss function as given in the table below.

	$\mathcal{A} \rightarrow$			
$\Theta \downarrow$	a_1	a_2	a_3	a_4
θ_1	1	1	2	2
θ_2	0	1	0	1

   Suppose the random variable takes value 0 with probability 1 for each θ.

   (i) Plot the risk set.

   (ii) Find a minimax rule. Is it unique ?

   (iii) Find a Bayes'rule w.r.t. the prior π satisfying $\pi(\theta_1) = 1$. Is it unique ? Is it admissible ?

   (iv) Find a Bayes'rule w.r.t. the prior π satisfying $\pi(\theta_1) = 1/2$.

   (c) Suppose Θ is finite and that a Bayes'rule δ_0 w.r.t a prior π exists. Show that if $\pi(\theta_) > 0, \forall \theta \in \Theta$ then δ_0 is admissible.

   Comment on your answer to the last part of Q(c) (iii) in view of this result.

   [3 x 3 + (5 + (4 + 1) + (4 + 1 + 3) + 3) + (6 + 2) = 38]

6. (a) Let Π denote the class of all priors on Θ. Show that for every decision rule δ.

$$\sup_{\pi \in \Pi} r(\pi, \delta) = \sup_{\theta \in \Theta} R(\theta, \delta).$$

(b) Consider a decision problem with finite Θ such that the risk set is bounded from below. Let D denote the class of all randomised decision rules.

(i) Show that the following relation hold and there exists a least favorable distribution π_0.

$$\inf_{\delta \in D} \sup_{\pi \in \Pi} r(\pi, \delta) = \sup_{\pi \in \Pi} \inf_{\delta \in D} r(\pi, \delta) = V.$$

(ii) Suppose further the risk set is closed from below. Then show that there exists a minimax rule δ_0, which is Bayes'w.r.t. π_0. [7 + (9 + 12) = 28]

7. Consider the problem $\Theta = \mathcal{A} = \{0, 1\}$ and the loss function as follows.

$$L(0,0) = L(1,1) = 0, \; L(0,1) = L(1,0) = 1.$$

The observed random variable follows the probability distribution

$$P(X = x|\theta) = 2^{-k}, \text{ if } x = k - \theta, \; k = 1, 2, \cdots.$$

(a) Describe the set of all nonrandomised decision rules.

(b) Plot the risk set in the plane.

(c) Find a minimax rule. [5 + 8 + 5 =18]